

ON NEWTON INTERPOLATION OF SYMMETRIC FUNCTIONS. A CHARACTERIZATION OF INTERPOLATION MACDONALD POLYNOMIALS.

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CONTENTS

1. Brief introduction
2. General interpolation problem
3. Examples of interpolation polynomials
 - 3.1 Universal interpolation polynomials
 - 3.2 Factorial monomial symmetric functions
 - 3.3 Factorial Schur functions
 - 3.4 Interpolation Macdonald polynomials
4. Statement of the characterization theorem
5. Reductions of the proof
6. Proof of the theorem in the two variables case ($n = 1$)
7. Proof of the theorem for $n > 1$
- A. Appendix. Table of perfect grids
- B. Appendix. An example of the universal interpolation polynomial

1. BRIEF INTRODUCTION

The purpose of this paper is to characterize interpolation Macdonald polynomials inside a very general Newton interpolation scheme for symmetric polynomials. This general Newton interpolation problem is discussed in Section 2; it depends as on a parameter on a map Ω

$$(\mathbb{Z}_{\geq 0})^{\# \text{ of variables}} \xrightarrow{\Omega} \text{ground field } \mathbb{k},$$

which we call a *grid* in \mathbb{k} . Our present understanding of this general problem can be described as follows: most of it is covered by an unexplored and mysterious ocean

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formed by generic grids Ω (see Section 3.1). In the midst of this abyss there are 3 pieces of dry land, namely the 3 following exactly solvable cases:

- (1) factorial monomial symmetric functions,
- (2) factorial Schur functions,
- (3) interpolation Macdonald polynomials,

which are described in Sections 3.2–3.4. The first case is just dull, the second one is still rather elementary; both of them are parameterized by an arbitrary sequence of pairwise distinct elements of the ground field \mathbb{k} . These two continents are joined by a beautiful archipelago of interpolation Macdonald polynomials. More precisely, by interpolation Macdonald polynomials we mean the so called *BC*-type interpolation Macdonald polynomials, introduced and studied in [Ok4]. As particular cases and degenerations these polynomials include polynomials studied by F. Knop, G. Olshanski, S. Sahi, and the author in a long series of papers, see References. These polynomials depends on 5 parameters of which only 3 are non-trivial because of an action of a 2-dimensional group of affine transformations. Some of the properties of these most remarkable polynomials are discussed in Section 3.4.

It is natural to ask if any simple abstract property characterizes the 3 above exactly solvable cases of our general interpolation problem. As such a property we propose the extra vanishing property (4.2) which says that the Newton interpolation polynomials should vanish not only at those points where they are supposed to vanish by their definition but also at certain extra points “for free”. More precisely, the polynomial labeled by a partition μ vanishes at the point labeled by a partition λ unless μ is less or equal to λ in the partial order of partitions by inclusion $\mu \subset \lambda$. We call all grids that enjoy this property *perfect*.

This extra vanishing property can be compared to the following well-known property of ordinary Macdonald polynomials. Although the Gram-Schmidt orthogonalization process requires a choice of a total order on the polynomials to be orthogonalized, the Macdonald orthogonal polynomials do not actually depend on the choice of a total order on the monomial symmetric functions as long this total order is compatible with the partial dominant order of partitions.¹

Also, the extra vanishing property can be compared to a well known phenomenon in integrable systems where many exactly solvable systems have “extra” integrals of motion, that is, more integrals of motion than is required by the definition of integrability [Kr,CV]. It is interesting to notice that certain integrable many-body systems to which interpolation Macdonald polynomials are very closely connected were conjectured to be characterized by this “extra” integrability property [CV]; later, however, certain new examples were found in [CFV].

Our situation is, of course, much simpler. Our main result (Section 4) is that the three above cases plus degenerations of the third one exhaust the set of all perfect grids. As a corollary of this theorem, we conclude that no other grid Ω admits a

¹It would be probably interesting to describe all interpolation or orthogonal symmetric polynomials which satisfy such a “extra triangularity” condition.

tableaux sum formula of the form (4.1) for the Newton interpolation polynomials, nor does it admit an integral representation of interpolation polynomials analogous to the q -integral representation for interpolation Macdonald polynomials obtained in [Ok4]. That is, any new exactly solvable case of symmetric Newton interpolation has to be based on some entirely new type of formulas.

The proof of this characterization theorem is given in Sections 5–7. Section 5 contains some general statements, whereas the two other sections are devoted to the consideration of the many possible cases.

There exist also non-symmetric Macdonald interpolation polynomials which form a linear basis in the algebra of all polynomials, see [Kn,S2]. It is plausible that those polynomials might have a similar characterization. Note also that a certain characterization of ordinary Macdonald polynomials inside some general class of orthogonal polynomials was found by S. Kerov in [K].

2. GENERAL INTERPOLATION PROBLEM

We consider Newton interpolation of symmetric polynomials in $n + 1$ variables ($n = 0, 1, \dots, \infty$) with coefficients in some infinite field \mathbb{k} . First assume for simplicity that $n < \infty$; the case $n = \infty$ will be covered at the end of this section. Any natural basis in the space of symmetric polynomials of degree $\leq d$ is indexed by partitions²

$$\lambda = (\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_n \geq 0)$$

of length $\ell(\lambda) \leq n + 1$ (recall that $\ell(\lambda)$ is the number of non-zero parts in λ) such that

$$|\lambda| \leq d.$$

Here $|\lambda| = \lambda_0 + \dots + \lambda_n$. Therefore, the knots of our Newton interpolation should be also indexed by partitions. In other words, we need a function

$$\mathcal{U} : \{\lambda, \ell(\lambda) \leq n + 1\} \longrightarrow \mathbb{k}^{n+1},$$

which takes a partition λ to the corresponding knot of interpolation $\mathcal{U}(\lambda)$. We can construct such a function in the following way. Choose a function

$$(2.1) \quad \Omega : \{0, \dots, n\} \times \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{k},$$

which we shall call a *grid* in \mathbb{k} , and then set

$$\mathcal{U}(\lambda) := (\Omega(0, \lambda_0), \dots, \Omega(n, \lambda_n)) \in \mathbb{k}^{n+1}.$$

To simplify notation, we shall use abbreviations

$$[ij] := \Omega(i, j), \quad \widehat{\lambda} := \mathcal{U}(\lambda).$$

²Our notation conventions about the partitions are slightly different from the standard ones used in [M1]. For example, we number the parts of partitions as well as coordinates of any vector starting from zero. This leads also to a different definition of a diagram of a partition but makes certain formulas look more symmetric.

Definition 2.1. A grid Ω is said to be *non-degenerate* if for any partition μ of length $\leq n + 1$ there exists a symmetric polynomial

$$P_\mu(x_0, \dots, x_n; \Omega) \in \mathbb{k}[x_0, \dots, x_n]^{S(n+1)}$$

satisfying the following Newton interpolation conditions

- (1) the degree of $P_\mu(x; \Omega)$ is $\leq |\mu|$;
- (2) $P_\mu(\hat{\lambda}; \Omega) = 0$ for all partitions λ of length $\leq n + 1$ such that $|\lambda| \leq |\mu|$ and $\lambda \neq \mu$;
- (3) $P_\mu(\hat{\mu}; \Omega) \neq 0$.

Remark 2.2. It is clear that if the grid Ω is non-degenerate then

- (1) all polynomials $P_\mu(x; \Omega)$ are uniquely defined up to a non-zero factor from the field \mathbb{k} ;
- (2) the polynomials $P_\mu(x; \Omega)$ as μ ranges over all partitions of length $\leq n + 1$ form a linear basis of the vector space $\mathbb{k}[x_0, \dots, x_n]^{S(n+1)}$ of all symmetric polynomials;
- (3) the degree of $P_\mu(x; \Omega)$ is precisely $|\mu|$.

The Newton interpolation polynomials $P_\mu(x; \Omega)$ generalize polynomials considered by S. Sahi in [S1].

All non-degenerate grids admit a simple description.

Proposition 2.3. *A grid Ω is non-degenerate if and only if*

$$(2.2) \quad [ij] \neq [i'j'], \quad \forall i, j \quad i \geq i', j < j'.$$

Before we prove this proposition let us introduce the following operation on grids (2.1). For any $m = 0, \dots, n$ we can define a grid

$$\Omega_m : \{0, \dots, m\} \times \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{k}$$

by simply restriction $\Omega_m(i, j) := \Omega(i, j)$ of the grid Ω .

Now the following proposition follows immediately from the definition of polynomials $P_\mu(x; \Omega)$. (Since so far these polynomials were defined up to a scalar factor only the equality (2.3) is to be understood for the moment as an equality up to a scalar factor. However, below in Definition 2.8 we shall choose a particular normalization of the polynomials $P_\mu(x; \Omega)$ which will make (2.3) into a precise equality.)

Proposition 2.4. *Suppose Ω is a non-degenerate grid and μ is a partition such that $\mu_n = 0$. Then*

$$(2.3) \quad P_\mu(x_0, \dots, x_{n-1}, [n0]; \Omega) = P_{(\mu_0, \dots, \mu_{n-1})}(x_0, \dots, x_{n-1}; \Omega_{n-1}).$$

Corollary 2.5. *If a grid Ω is non-degenerate then so are all grids Ω_m , $m = 0, \dots, n$.*

Proof of Proposition 2.3. Suppose that Ω is non-degenerate and show that

$$[ij] \neq [i'j'], \quad i \geq i', j < j'.$$

By the above Corollary we can assume that $i = n$. Suppose that, on the contrary, $[nj] = [i'j']$ and $j' > j$. Then the following polynomial

$$\prod_{m=0}^n \prod_{k=0}^{j'-1} (x_m - [n, k])$$

is symmetric, has degree $n j'$ and vanishes at all points $\widehat{\lambda}$ such that $|\lambda| \leq nj'$. Thus, the “only if” part of the proposition is established.

The “if” part of the proposition follows from the following argument, which is an abstract form of the argument of S. Sahi, see [S1].

Proposition 2.6. *Suppose a grid Ω satisfies the conditions (2.2) and let $\mathbb{S} \subset \mathbb{k}$ be any ring such that*

$$[ij], \frac{1}{[ij] - [i'j']} \in \mathbb{S}, \quad i \geq i', j < j'.$$

Then for any $d = 0, 1, \dots$ and any function

$$(2.4) \quad \phi : \{ \lambda, |\lambda| \leq d, \ell(\lambda) \leq n+1 \} \longrightarrow \mathbb{S}$$

there exists unique symmetric polynomial

$$f_\phi \in \mathbb{S}[x_0, \dots, x_n]^{S(n+1)}$$

of degree d such that

$$(2.5) \quad f_\phi(\widehat{\lambda}) = \phi(\lambda).$$

Proof. Induct on n and d . The cases $n = 0$ or $d = 0$ are clear. Suppose $n, d > 0$. Introduce a linear map

$$\text{ext} : \mathbb{S}[x_0, \dots, x_{n-1}]^{S(n)} \longrightarrow \mathbb{S}[x_0, \dots, x_n]^{S(n+1)},$$

which, by definition, takes a monomial symmetric function in variables

$$x_0 - [n0], \dots, x_{n-1} - [n0]$$

to the same monomial symmetric function in variables

$$x_0 - [n0], \dots, x_n - [n0].$$

Observe that it is a degree preserving injection and that

$$(2.6) \quad (\text{ext } f)(x_0, \dots, x_{n-1}, [n0]) = f(x_0, \dots, x_{n-1}).$$

Now given ϕ we shall look for the solution f_ϕ in the form

$$(2.7) \quad f_\phi = \text{ext } f_1 + f_2 \prod_{i=0}^n (x_i - [n0]),$$

where f_1 and f_2 are unknown polynomials such that

$$\begin{aligned} f_1 &\in \mathbb{S}[x_0, \dots, x_{n-1}]^{S(n)}, & \deg f_1 &= d, \\ f_2 &\in \mathbb{S}[x_0, \dots, x_n]^{S(n+1)}, & \deg f_2 &= d - n. \end{aligned}$$

First, consider the equations (2.5) for partitions λ such that $\lambda_n = 0$. Since then the second summand in (2.7) vanishes these equations by (2.6) and inductive assumption determine the polynomial f_1 .

Now consider the equations (2.5) for the remaining partitions λ (that is, for λ such that $\lambda_n > 0$). Rewrite them in the form

$$(2.8) \quad f_2(\widehat{\lambda}) = \frac{\phi(\lambda) - (\text{ext } f_1)(\widehat{\lambda})}{\prod_i ([i\lambda_i] - [n0])}, \quad \lambda_n > 0,$$

and observe that the RHS of (2.8) lies in \mathbb{S} . The set of partitions λ such that $|\lambda| \leq d$ and $\lambda_n > 0$ is in bijection

$$\lambda \mapsto \lambda - \bar{1} = (\lambda_0 - 1, \dots, \lambda_n - 1)$$

with the set of partitions such that $|\lambda| \leq d - n$. Therefore, replacing Ω by Ω^1 and using the inductive hypothesis we can find a polynomial f_2 of degree $d - n$ satisfying the equations (2.8). This proves existence of the required polynomial f_ϕ .

To verify uniqueness of f_ϕ it suffices to consider the case $\mathbb{S} = \mathbb{k}$. We have an obvious \mathbb{k} -linear map from the space of symmetric polynomials f of degree $\leq d$ to the space of functions ϕ of the form (2.4), namely

$$f \mapsto \phi, \quad \phi(\lambda) := f(\widehat{\lambda}).$$

Both spaces have same dimension and we just have proved that this map is surjective. Hence it is an isomorphism with the inverse map

$$\phi \mapsto f_\phi.$$

This concludes the proof of the Propositions 2.6 and 2.3. \square

Let us introduce another operation on grids (2.1). Given a grid Ω , we can for any $k = 0, 1, 2, \dots$ define a new grid Ω^k

$$\Omega^k : \{0, \dots, n\} \times \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{k}$$

by the formula

$$\Omega^k(i, j) := \Omega(i, j + k).$$

Then the following property follows immediately from Definition 2.1 and Proposition 2.3 (As in the case of Proposition 2.4, observe that since the normalization of the polynomials $P_\mu(x; \Omega)$ is yet to be specified the equality (2.9) is to be understood for the moment as an equality up to a scalar factor.)

Proposition 2.7. *Suppose Ω is a non-degenerate grid. Then the grid Ω^1 is also non-degenerate and for any partition μ such that $\mu_n > 0$ we have*

$$(2.9) \quad P_\mu(x_0, \dots, x_n; \Omega) = P_{\mu - \bar{1}}(x_0, \dots, x_n; \Omega^1) \prod_{i=0}^n (x_i - [n0]),$$

where $\mu - \bar{1}$ stands for partition $(\mu_0 - 1, \dots, \mu_n - 1)$.

Now suppose that Ω is a non-degenerate grid and let us specify the normalization of the interpolation polynomials $P_\mu(x; \Omega)$. Let us identify any partition λ with its diagram which is, by definition, the following subset of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$

$$\lambda = \{(i, j), j \leq \lambda_j - 1\} \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}.$$

Note that this definition differs by a coordinate shift by 1 from the standard definition used in [M1]. Let λ' stand for the partition corresponding to the transposed diagram of λ , that is

$$(2.10) \quad \lambda'_j := \#\{i, \lambda_i > j\}.$$

We are free to normalize the polynomials $P_\mu(x; \Omega)$ by setting their value at the point $\hat{\mu}$ to any non-zero element of \mathbb{k} . We make the following choice

Definition 2.8. Suppose Ω is a non-degenerate grid and μ is a partition with $\leq n + 1$ parts. Then we define $P_\mu(x; \Omega)$ to be the unique polynomial satisfying the conditions of Definition 2.1 and the following normalization condition:

$$(2.11) \quad P_\mu(\hat{\mu}; \Omega) = \prod_{(i,j) \in \mu} ([i, \mu_i] - [\mu'_j - 1, j]).$$

The reason we prefer the normalization (2.11) is the following

Proposition 2.9. *The normalization (2.11) is the unique normalization compatible with (2.3) and (2.9).*

Proof. Induct on n and $|\mu|$. Since for any partition μ either $\mu_n > 0$ or $\mu_n = 0$ we can apply either (2.3) or (2.9) to the evaluation of

$$P_\mu(\widehat{\mu}; \Omega)$$

and thus reduce it to the case of smaller values of $|\mu|$ or n respectively. \square

By definition, let $\mathbb{A} \subset \mathbb{k}$ be the subring generated by the elements

$$(2.12) \quad \mathbb{A} = \mathbb{Z} \left[[ij], \frac{1}{[ij] - [i'j']} \right] \subset \mathbb{k}, \quad i \geq i', j < j'.$$

Then since by (2.11) we have

$$P_\mu(\widehat{\mu}; \Omega) \in \mathbb{A}$$

for any μ we conclude from Proposition 2.6 and Definition 2.8 that

Proposition 2.10. $P_\mu(x; \Omega) \in \mathbb{A}[x_0, \dots, x_n]^{S(n+1)}$.

Remark 2.11. Now let us explain the definition of the polynomials $P_\mu(x; \Omega)$ in the case of infinitely many variables ($n = \infty$). Suppose we are given a grid

$$(2.13) \quad \Omega : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{k}.$$

Then we need first a suitable definition of the algebra of symmetric polynomials in infinitely many variables. For any $m = 0, 1, 2, \dots$ define a homomorphism

$$(2.14) \quad \text{Res}_m^{m+1} : \mathbb{k}[x_0, \dots, x_{m+1}]^{S(m+2)} \longrightarrow \mathbb{k}[x_0, \dots, x_m]^{S(m+1)}$$

by the formula

$$(\text{Res}_m^{m+1} f)(x_0, \dots, x_m) = f(x_0, \dots, x_m, [m+1, 0]).$$

Observe that

$$\deg \text{Res}_m^{m+1} f \leq \deg f.$$

By definition, let Λ^Ω be

$$\Lambda^\Omega := \varprojlim \mathbb{k}[x_0, \dots, x_m]^{S(m+1)}$$

the inverse limit of filtered (by the degree of polynomials) algebras with respect to homomorphisms (2.14). Observe that by construction for any

$$f \in \Lambda^\Omega$$

its degree $\deg f$ is well defined and so are its values at the points of the form

$$f(\widehat{\lambda}) = f([0\lambda_0], [1\lambda_1], [2\lambda_2], \dots),$$

where λ is a partition. Therefore the interpolation problem described in Definition 2.1 is well defined for polynomials in the algebra Λ^Ω , so it makes sense to ask whether the grid Ω is non-degenerate and what are the corresponding Newton interpolation polynomials. Using definitions and Propositions 2.3 and 2.4 one easily checks that

- (1) a grid (2.13) is non-degenerate if and only if the conditions (2.2) are satisfied (which is also equivalent to non-degeneracy of all grids Ω_m , $0 \leq m < \infty$);
- (2) for any partition μ the sequence

$$\left\{ P_{(\mu_0, \dots, \mu_m)}(x_0, \dots, x_m; \Omega_m) \right\}_{m \geq \ell(\mu)}$$

defines an element of Λ^Ω which equals $P_\mu(x; \Omega)$.

3. EXAMPLES OF INTERPOLATION POLYNOMIALS

3.1 Universal interpolation polynomials.

Let \mathbb{k}^u be the field

$$\mathbb{k}^u = \mathbb{Q}(u_{ij}), \quad (i, j) \in \{0, \dots, n\} \times \mathbb{Z}_{\geq 0},$$

of rational functions in variables u_{ij} and let the grid Ω^u be the grid

$$[ij]^u = u_{ij},$$

which is clearly non-degenerate (the superscript u stands here for “universal”). We have the corresponding polynomials

$$(3.1) \quad P_\mu(x; \Omega^u) \in \mathbb{A}^u[x_0, \dots, x_n]^{S(n+1)},$$

where the ring \mathbb{A}^u is, according to the definition (2.12), the ring

$$\mathbb{A}^u = \mathbb{Z} \left[u_{ij}, \frac{1}{u_{ij} - u_{i'j'}} \right] \subset \mathbb{k}^u, \quad i \geq i', j < j'.$$

The polynomials (3.1) are universal in the following sense. For any field \mathbb{k} and any non-degenerate grid

$$\Omega : \{0, \dots, n\} \times \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{k}$$

we have a natural homomorphism of rings

$$\psi : \mathbb{A}^u \rightarrow \mathbb{k}, \quad \psi(u_{ij}) = [ij]$$

which is well defined by non-degeneracy of Ω and takes universal interpolation polynomials to the corresponding specific ones

$$\psi(P_\mu(x; \Omega^u)) = P_\mu(x; \Omega).$$

One can compute a few universal polynomials explicitly (see an Example in Appendix B) but the formulas get very soon very complicated. One is forced therefore to look at least general but nicer examples.

The three other examples we shall consider in this section will be *exactly solvable* in the following sense: there exist rather simple closed formulas for the interpolation polynomials. For simplicity we consider the case of finitely many variables ($n < \infty$). However, all formulas are stable in the sense described in Remark 2.11.

3.2 Factorial monomial symmetric functions.

Suppose that

$$[ij] = c_j,$$

where c_0, c_1, \dots are pairwise distinct elements of \mathbb{k} . Then it is easy to see that the interpolation polynomials P_μ are simply factorial monomial symmetric functions

$$(3.2) \quad P_\mu(x; \Omega) = \frac{1}{\# \text{stab}_{S(n+1)} \mu} \sum_{s \in S(n+1)} \prod_{i=0}^n \prod_{j=0}^{\mu_i-1} (x_{s(i)} - c_j),$$

where $\# \text{stab}_{S(n+1)} \mu$ is the number of permutations $s \in S(n+1)$ which leave the vector $\mu = (\mu_0, \dots, \mu_n)$ invariant. It is also easy to see that the polynomials (3.2) vanish at more more point then it is prescribed by their definition. Namely, we have

$$P_\mu(\widehat{\lambda}; \Omega) = 0, \quad \text{unless } \mu \subset \lambda,$$

where the notation $\mu \subset \lambda$ means that $\mu_i \leq \lambda_i$ for all i (which is equivalent to the diagram of μ being a subset of the diagram of λ).

3.3 Factorial Schur functions.

Suppose that

$$[ij] = c_{j-i},$$

where $\dots, c_{-1}, c_0, c_1, \dots$ are pairwise distinct elements of \mathbb{k} . Introduce the following factorial Schur polynomial (see [M2] and references therein and also [OO1])

$$(3.3) \quad s_\mu(x; \Omega) := \frac{\det [(x_i - c_{-n}) \cdots (x_i - c_{\mu_j - j - 1})]_{0 \leq i, j \leq n}}{\prod_{0 \leq i < j \leq n} (x_i - x_j)}.$$

First observe that (3.3) is indeed a polynomial because the numerator in (3.3) is an anti-symmetric polynomial in x_0, \dots, x_n and hence is divisible by the denominator. It is also clear that the ratio has degree $|\mu|$ and its top-degree component is the classical Schur function $s_\mu(x)$.

We shall check momentarily (borrowing the argument from [Ok1], Section 2.4; see also [OO1]) that we have

$$(3.4) \quad P_\mu(x; \Omega) = s_\mu(x; \Omega).$$

First show that

$$(3.5) \quad s_\mu(\widehat{\lambda}; \Omega) = 0, \quad \text{unless } \mu \subset \lambda.$$

Observe that the denominator in (3.3) does not vanish at any point of the form $\widehat{\lambda}$. Therefore, it suffices to check that the numerator vanishes unless $\mu \subset \lambda$. Suppose that $\lambda_k < \mu_k$ for some k . Then for all $0 \leq j \leq k \leq i \leq n$ we have

$$\lambda_i \leq \lambda_k < \mu_k \leq \mu_j$$

and the corresponding matrix element vanishes

$$(c_{\lambda_i-i} - c_{-n}) \cdots (c_{\lambda_i-i} - c_{\mu_j-j-1}) = 0.$$

Then the matrix

$$(3.6) \quad [(c_{\lambda_i-i} - c_{-n}) \cdots (c_{\lambda_i-i} - c_{\mu_j-j-1})]_{0 \leq i, j \leq n}$$

has a block-triangular form with index sets $\{0, \dots, k-1\}$, $\{k\}$, and $\{k+1, \dots, n\}$. The middle diagonal block is zero and so is the determinant of (3.6). This proves (3.5).

Now let us compute $s_\mu(\widehat{\mu}; \Omega)$. By the same argument the matrix (3.6) with $\lambda = \mu$ is triangular, therefore

$$\begin{aligned} s_\mu(\widehat{\mu}; \Omega) &= \frac{\prod_{i=0}^n (c_{\mu_i-i} - c_{-n}) \cdots (c_{\mu_i-i} - c_{\mu_i-i-1})}{\prod_{0 \leq i < j \leq n} (c_{\mu_i-i} - c_{\mu_j-j})} \\ &= \prod_{(i,j) \in \mu} (c_{\mu_i-i} - c_{\mu'_j-j-1}), \end{aligned}$$

where one goes from the first line to the second by a standard combinatorial argument (used in e.g. proof of the hook-length formula). This proves (3.4).

There is a convenient tableaux sum formula (see [M2] and [GG]) for the polynomial (3.3). Given a partition μ , a *reverse* tableaux T on μ is, by definition, a function on the diagram of μ

$$\mu \ni (i, j) \xrightarrow{T} \{0, \dots, n\},$$

that decreases along the columns

$$T(i, j) > T(i', j), \quad i > i',$$

and does not increase along the rows

$$T(i, j) \geq T(i, j'), \quad j < j'.$$

With this notation the tableaux sum formula is

$$(3.7) \quad s_\mu(x; \Omega) = \sum_T \prod_{(i,j) \in \mu} (x_{T(i,j)} - c_{j-i-T(i,j)}),$$

where the summation ranges over all reverse tableaux T on μ .

3.4 Interpolation Macdonald polynomials.

These polynomials depend on 5 parameters a, b, c, q, t from³ the field \mathbb{k} such that $q, t \neq 0$ and the corresponding grid is given by

$$(3.8) \quad [ij] = a + bq^j t^i + \frac{c}{q^j t^i} \in \mathbb{k}.$$

The non-degeneracy of such a grid is assured by inequalities

$$\begin{aligned} q^k &\neq t^l, \quad k > 0, l \geq 0, \\ bq^k t^l &\neq c, \quad k > 0, l > 0, \end{aligned}$$

see more precise conditions in Appendix A. The simultaneous shift & scale transformations

$$x_i \mapsto C_1 x_i + C_2, \quad i = 0, \dots, n, \quad C_1 \in \mathbb{k} \setminus 0, \quad C_2 \in \mathbb{k},$$

reduce the number of non-trivial parameters to following 3

$$q, t, c/b,$$

where the last one can also assume the value ∞ .

In full generality the corresponding interpolation polynomials $P_\mu(x; \Omega)$ were considered in [Ok4]. Important particular cases and degenerations of them were considered earlier by F. Knop, G. Olshanski, S. Sahi, and the author in a long series of papers, see References.

The following tableaux sum formula for the polynomials $P_\mu(x; \Omega)$ follows by a change of variables from the statement of Theorem 5.2 in [Ok4]

$$(3.9) \quad P_\mu(x; \Omega) = \sum_T \psi_T(q, t^{-1}) \prod_{(i,j) \in \mu} (x_{T(i,j)} - [i + T(i, j), j]),$$

where the summation ranges over all reverse tableaux T on μ and $\psi_T(q, t^{-1})$ is a certain two-parametric weight of a tableaux T introduced by I. Macdonald in [M1], section VI.7. This weight is a product of factors of the form

$$(1 - q^k t^l), \quad k, -l \in \mathbb{Z}_{\geq 0},$$

and it appears in the tableaux sum formula for the ordinary Macdonald polynomials

$$(3.10) \quad P_\mu(x; q, t^{-1}) = \sum_T \psi_T(q, t^{-1}) \prod_{(i,j) \in \mu} x_{T(i,j)}.$$

³Actually, the parameters a, b, c, q, t may lie in some extension of the field \mathbb{k} ; then (3.8) implies that q and t satisfy quadratic equations with coefficients in \mathbb{k}

In (3.10), the LHS denotes the ordinary Macdonald polynomial with parameters q and t^{-1} (we use Macdonald's notation for it; it is not to be confused with our interpolation polynomials). Note that, in particular, (3.10) is the highest degree term of (3.9).

The 5-parametric grid (3.8) admits many degenerations; they are all listed in Appendix A. As in the two previous examples, the polynomials $P_\mu(x; \Omega)$ also enjoy the property

$$P_\mu(\widehat{\lambda}; \Omega) = 0, \quad \text{unless } \mu \subset \lambda,$$

which can be seen from (3.9) (see Lemma 4.1 below) but is actually used in the proof of (3.9).

The polynomials (3.9) seem to be very distinguished special polynomials. They have a wealth of applications in various fields of mathematics, see e.g. [Ok1, OO4-5, KOO, Ok5].

As to the practical interpolation, in the particular case when

$$bc = 0$$

there exists a nice efficient algorithm for Newton interpolation, see [Ok3]. It really speeds up for the following (Jack) degeneration of the grid (3.8)

$$[ij] = \alpha + \beta j + \beta' i.$$

A remarkable feature of that algorithm is that the coefficients of the Newton interpolation expansion of any symmetric polynomial can be found without actually computing these (rather complicated) Newton interpolation polynomials. That algorithm can be also used for expansion in ordinary Macdonald and Jack polynomials.

Note that in two particular cases

$$t = 1 \quad \text{or} \quad t = 1/q$$

the number (3.8) depends only on j or $j - i$ respectively. Hence for these values of t the interpolation Macdonald polynomials become particular cases of factorial monomial symmetric functions or factorial Schur functions.

4. STATEMENT OF THE CHARACTERIZATION THEOREM

The exactly solvable examples 2–4 of the previous section have at least two following common features. The interpolation polynomials admit a tableaux sum formula of the form⁴

$$(4.1) \quad P_\mu(x; \Omega) = \sum_T \text{weight}(T) \prod_{(i,j) \in \mu} (x_{T(i,j)} - [i + T(i,j), j]),$$

⁴It is easy to see that the formula (3.2) can be written in the form (4.1) with $\text{weight}(T) \in \{0, 1\}$.

where the summation ranges over all reverse tableaux T on μ and

$$\text{weight}(T) \in \mathbb{k}$$

is some weight of a tableaux T . The interpolation polynomials also enjoy the following *extra vanishing* property

$$(4.2) \quad P_\mu(\widehat{\lambda}; \Omega) = 0, \quad \text{unless } \mu \subset \lambda.$$

In fact, (4.2) is a consequence of (4.1) as the following argument (borrowed from [Ok1], Section 3.8) shows

Lemma 4.1. *If T is a reverse tableau on μ and λ is a partition then*

$$\prod_{(i,j) \in \mu} ([T(i,j), \lambda_{T(i,j)}] - [i + T(i,j), j]) = 0,$$

unless $\mu \subset \lambda$.

Proof. Suppose that

$$[T(i,j), \lambda_{T(i,j)}] - [i + T(i,j), j] \neq 0, \quad \forall (i,j) \in \mu.$$

In particular, for $i = 0$ we obtain

$$(4.3) \quad \lambda_{T(0,0)} \neq 0, \lambda_{T(0,1)} \neq 1, \dots, \lambda_{T(0,j)} \neq j, \dots.$$

On the other hand, since T is a reverse tableau we have

$$(4.4) \quad \lambda_{T(0,0)} \leq \lambda_{T(0,1)} \leq \dots$$

The inequalities (4.3) and (4.4) imply that

$$\lambda_{T(0,j)} > j, \quad j = 0, 1, \dots.$$

Again, since T is a reverse tableaux we have

$$(4.5) \quad T(0,j) < T(1,j) < \dots < T(\mu'_j - 1, j)$$

and also

$$(4.6) \quad j < \lambda_{T(0,j)} \leq \lambda_{T(1,j)} \leq \dots \leq \lambda_{T(\mu'_j - 1, j)}.$$

By the definition (2.10) the inequalities (4.5) and (4.6) yield that

$$\lambda'_j \geq \mu'_j, \quad j = 0, 1, \dots,$$

which is equivalent to $\mu \subset \lambda$. \square

Remark 4.2. By employing the same argument as used in the proof of Theorem 5.1 in [Ok4] one can show *a priori* that, conversely, the extra vanishing (4.2) implies existence of a tableaux sum formula (4.1). We shall obtain this implication as a corollary of our main theorem.

Remark 4.3. It is also clear that the extra vanishing (4.2) follows immediately from any analog of the q -integral representation established in [Ok4] for the interpolation Macdonald polynomials in the case $a, b, c, q, t \in \mathbb{C}$ and $|q| < 1$.

The above discussion justifies the following

Definition 4.4. We shall call a non-degenerate grid Ω *perfect* if the polynomials $P_\mu(x; \Omega)$ enjoy the extra vanishing property (4.2).

Our main result is the following

Main Theorem. *The following is the list of all perfect grids Ω :*

E₁. $[ij] = c_j$, where c_0, c_1, \dots are pairwise distinct elements of \mathbb{k} ; the corresponding interpolation polynomials are the factorial monomial symmetric function (see Section 3.2).

E₂. $[ij] = c_{j-i}$, where $\dots, c_{-1}, c_0, c_1, \dots$ are pairwise distinct elements of \mathbb{k} ; the corresponding interpolation polynomials are the factorial Schur function (see Section 3.3).

I. $[ij] = a + bq^j t^i + cq^{-j} t^{-i}$, where a, b, c, q, t are elements of a certain extension of \mathbb{k} ; the corresponding interpolation polynomials are the interpolation Macdonald polynomials (see Section 3.4).

II-IV. The grid Ω and the interpolation polynomials are one of the degeneration of the previous case (see Appendix A).

Since the two first cases are much simpler than the remaining ones we refer to them as to the 1st and 2nd elementary cases and number them by E₁ and E₂.

It follows from the above theorem together with Lemma 4.1 and Remark 4.3 that if there exist any other exactly solvable cases of the general interpolation problem described in Section 2, then the formulas for the corresponding interpolation polynomials should have some entirely new structure.

The proof of the theorem will be given in Sections 5–7.

4. REDUCTIONS OF THE PROOF

First from (2.3) and (2.9) one immediately derives the following

Proposition 5.1. *If a grid Ω*

$$(5.1) \quad \Omega : \{0, \dots, n\} \times \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{k}$$

is perfect then so are all grids Ω^k , $k = 0, 1, \dots$, and Ω_m , $m = 0, \dots, n$.

Introduce one more operation on grids. Given a grid (5.1) we can define a grid

$$_l\Omega : \{0, \dots, n-l\} \longrightarrow \mathbb{k}, \quad l = 0, \dots, n,$$

by setting

$$_l\Omega(i, j) := \Omega(i+l, j).$$

From Proposition 2.3 it is clear that the operation

$$\Omega \longmapsto _l\Omega$$

preserves non-degeneracy. We now plan to show that it preserves perfectness as well. First, we establish the following

Proposition 5.2. *Let Ω be a perfect grid and let $P_\mu(x; \Omega)$ be the corresponding Newton interpolation polynomials. We have*

$$(5.2) \quad P_\mu(x; \Omega) = x^\mu + \dots,$$

where $x^\mu = x_0^{\mu_0} \cdots x_n^{\mu_n}$ and dots stand for lower monomials in lexicographic order.

Proof. First show that

$$(5.3) \quad P_\mu(x; \Omega) = c_\mu x^\mu + \dots, \quad c_\mu \in \mathbb{k},$$

for certain constants c_μ . Induct on $|\mu|$, the case $|\mu| = 0$ being clear. Let m

$$m := \deg_{x_0} P_\mu(x; \Omega)$$

be the degree of $P_\mu(x; \Omega)$ as of a polynomial in x_0 . If $m < \mu_0$ then (5.3) is established with $c_\mu = 0$.

Suppose therefore that $m \geq |\mu_0|$ and consider the leading coefficient

$$g(x_1, \dots, x_n) := [x_0^m] P_\mu(x; \Omega) \in \mathbb{k}[x_1, \dots, x_n]^{S(n)}$$

of $P_\mu(x; \Omega)$ as of a polynomial in x_0 . By our hypothesis g is a non-zero polynomial of degree

$$\deg g \leq |\mu| - \mu_0 = \mu_1 + \cdots + \mu_n.$$

We claim that

$$g([1, \lambda_0], \dots, [n\lambda_{n-1}]) = 0$$

for all partitions λ such that $\lambda_i < \mu_{i+1}$ for some $i = 0, \dots, n-1$. Indeed, by the extra vanishing condition (4.2) the polynomial

$$P_\mu(x_0, [1, \lambda_0], \dots, [n\lambda_{n-1}]; \Omega)$$

has in this case infinitely many zeros

$$x_0 = [0, \lambda_0], [0, \lambda_0 + 1], [0, \lambda_0 + 2], \dots$$

and hence vanishes identically. Since the grid ${}_1\Omega$ is non-degenerate we conclude that

$$g = \text{const } P_{(\mu_1, \dots, \mu_n)}(x_1, \dots, x_n; {}_1\Omega),$$

which by inductive assumption establishes (5.3).

Now recall that for any non-degenerate grid the polynomials $P_\mu(x; \Omega)$ form a linear basis in the \mathbb{k} -linear space of all symmetric polynomials. This immediately implies that

$$\forall \mu \quad c_\mu \neq 0.$$

Let us renormalize the polynomials $P_\mu(x; \Omega)$. Introduce new polynomials

$$\tilde{P}_\mu(x; \Omega) := \frac{1}{c_\mu} P_\mu(x; \Omega)$$

for which we have

$$\tilde{P}_\mu(x; \Omega) = x^\mu + \dots$$

It is clear that these polynomials also satisfy the equalities (2.3) and (2.9). Therefore, by Proposition 2.9 we conclude that

$$\tilde{P}_\mu(x; \Omega) = P_\mu(x; \Omega),$$

which concludes the proof of the proposition. \square

In fact, the above argument establishes more than just (5.2). We have also proved the two following facts:

Proposition 5.3. *We have*

$$P_\mu(x; \Omega) = x_0^{\mu_0} P_{(\mu_1, \dots, \mu_n)}(x_1, \dots, x_n; {}_1\Omega) + \dots,$$

where dots stand for terms of lower degree in x_0 .

Proposition 5.4. *If a grid Ω is perfect then so are the grids ${}_1\Omega, {}_2\Omega, \dots$*

Let us combine the 3 operations

$$\Omega \mapsto \Omega_m, \quad \Omega \mapsto \Omega^k, \quad \Omega \mapsto {}_l\Omega$$

as follows. Given a grid (5.1) we define for all $0 \leq l \leq m \leq n$ and all $k \geq 0$ a new grid

$${}_l\Omega_m^k : \{0, \dots, m-l\} \times \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{k}$$

by the formula

$${}_l\Omega_m^k(i, j) := \Omega(i+l, j+k).$$

Then by Propositions 5.1 and 5.4 we have

Proposition 5.5. *If a grid Ω is perfect then so are all grids ${}_l\Omega_m^k$.*

From now on we assume that Ω is a certain given perfect grid and our goal is to show that Ω is one of the grids listed in the Appendix A.

Introduce the following notation. By the symbol ε we shall denote *some non-zero* element of \mathbb{k}

$$\varepsilon \in \mathbb{k} \setminus \{0\}.$$

The purpose of this notation is to denote irrelevant overall factors in our equations. An example of an element ε is any product of factors of the form

$$[ij] - [i'j'], \quad j < j', i \geq i'.$$

Now we prove the following

Proposition 5.6. *Let F be the following rational function*

$$(5.4) \quad F(y_1, y_2, y_3, y_4, y_5) := \frac{y_4^2 - y_4y_1 - y_4y_5 + y_2y_3 + y_1y_5 - y_3^2}{y_2 - y_3}.$$

Then we have

$$(5.5) \quad [i+1, j+2] = F([i, j], [i+1, j], [i, j+1], [i+1, j+1], [i, j+2]),$$

for all j and all $i \leq n-1$. We also have

$$(5.6) \quad [i+2, j+1] = F([i, j], [i, j+1], [i+1, j], [i+1, j+1], [i+2, j]),$$

for all j and all $i \leq n-2$.

Proof. Prove (5.5). By virtue of Proposition 5.5 it suffices to establish (5.5) for

$$n = 1, \quad i = j = 0.$$

Consider the condition

$$(5.7) \quad P_{(3,0)}(\widehat{(2,2)}; \Omega) = 0.$$

A direct computation shows that

$$(5.8) \quad \begin{aligned} P_{(3,0)}(\widehat{(2,2)}; \Omega) &= \\ &= \varepsilon([00][02] - [00][11] - [01]^2 + [01][10] + [01][12] - [02][11] - [10][12] + [11]^2), \end{aligned}$$

where

$$\varepsilon = \frac{([12] - [11])([12] - [10])}{[01] - [10]}.$$

Since

$$[01] - [10] \neq 0$$

we deduce from (5.7) that

$$(5.9) \quad [12] = F([00], [10], [01], [11], [02]).$$

This proves (5.5).

Prove (5.6). By virtue of Proposition 5.5 it suffices to establish it for

$$n = 2, \quad i = j = 0.$$

Consider the condition

$$P_{(2,0,0)}(\widehat{(1,1,1)}; \Omega) = 0.$$

A direct computation shows that

$$(5.10) \quad P_{(2,0,0)}(\widehat{(1,1,1)}; \Omega) = \varepsilon([10][21] - [01][21] + [11]^2 - [11][00] + [01][10] - [10]^2 + [00][20] - [11][20])$$

where

$$\varepsilon = \frac{[21] - [20]}{[10] - [01]}.$$

Therefore,

$$(5.11) \quad [21] = F([00], [01], [10], [11], [20]).$$

This concludes the proof. \square

It is clear from the formula (5.4) and non-degeneracy that the equalities (5.5) and (5.6) can be reversed as follows

Corollary 5.7. *We have*

$$(5.12) \quad [ij] = F([i+1, j+2], [i, j+2], [i+1, j+1], [i, j+1], [i+1, j]),$$

for all j and $i \leq n-1$, and also

$$(5.13) \quad [ij] = F([i+2, j+1], [i+2, j], [i+1, j+1], [i+1, j], [i, j+1]),$$

for all j and $i \leq n-2$.

The above equalities immediately result in the following two propositions.

Proposition 5.8. *Suppose that for some i and j we have*

$$[i, j] = [i+1, j+1].$$

Then Ω is a grid of type E_2 that is, $[i, j]$ depends on $j-i$ only.

Proof. Follows from (5.5), (5.6), (5.12), (5.13), and the following identity

$$(5.14) \quad F(z, u, v, z, w) = v. \quad \square$$

Proposition 5.9. *Suppose that $n=1$ and for some j we have*

$$[0, j] = [1, j], \quad [0, j+1] = [1, j+1].$$

Then Ω is a grid of type E_1 that is, for all $k \geq 0$ we have

$$[0, k] = [1, k].$$

Proof. Follows from (5.5), (5.12), and the following identity

$$(5.15) \quad F(u, u, v, v, w) = w. \quad \square$$

6. PROOF OF THE THEOREM IN THE TWO VARIABLES ($n = 1$) CASE.

Consider the condition

$$P_{(4,0)}(\widehat{(3,2)}; \Omega) = 0$$

and make the substitution (5.11). After that substitution one computes:

$$(6.1) \quad P_{(4,0)}(\widehat{(3,2)}; \Omega) = \varepsilon ([00] - [11]) F_1([00], [10], [01], [11], [02], [03]),$$

where ε is the following invertible element

$$\varepsilon = \frac{([03] - [02])([12] - [11])([12] - [10])}{([02] - [10])([10] - [01])^2}$$

and F_1 denotes the following polynomial

$$F_1(y_1, y_2, y_3, y_4, y_5, y_6) := y_6 G_1(y_2, y_3, y_4) - G_0(y_1, y_2, y_3, y_4, y_5),$$

where

$$\begin{aligned} G_0(y_1, y_2, y_3, y_4, y_5) := & y_1 y_4 y_2 - y_1 y_4 y_5 - y_1 y_2 y_5 + y_1 y_5^2 + y_3^2 y_4 + y_3^2 y_2 \\ & - y_3^2 y_5 - y_3 y_4^2 - y_3 y_4 y_2 - y_3 y_2^2 + y_3 y_5^2 + y_4^2 y_5 + y_4 y_2 y_5 - y_4 y_5^2 + y_2 y_5^2 - y_2 y_5 \end{aligned}$$

and

$$G_1(y_2, y_3, y_4) := (y_3 - y_2)(y_3 - y_4).$$

Since the product (6.1) vanishes we have the following alternatives.

Case 1. We have $[00] - [11] = 0$. Then by Corollary we find ourselves in the E_2 case.

Case 2. We have

$$(6.2) \quad [03] G_1([10], [01], [11]) - G_0([00], [10], [01], [11], [02]) = 0.$$

Then, since the above equation is linear in $[03]$, we have again two possibilities.

Case 2.1. We have

$$(6.3) \quad G_0([00], [10], [01], [11], [02]) = G_1([10], [01], [11]) = 0.$$

Then the only solution of

$$G_1([10], [01], [11]) = ([01] - [10])([01] - [11]) = 0$$

compatible with the non-degeneracy is

$$(6.4) \quad [01] = [11].$$

Substituting it into the first equation we obtain

$$([02] - [10])([02] - [01])([00] - [10]) = 0.$$

From non-degeneracy we obtain

$$[00] = [10].$$

Thus, by Proposition 5.9 we are in the E_1 case.

Case 2.2. We assume that the grid Ω is not of the elementary types E_1 or E_2 . Then we can uniquely determine $[03]$ from the values of $[00]$, $[10]$, $[01]$, $[11]$, and $[02]$ by the formula

$$[03] = \frac{G_0([00], [10], [01], [11], [02])}{G_1([10], [01], [11])}.$$

Then using (5.5) we can also uniquely determine $[13]$. We claim that we can determine $[04]$ as well. To that end we repeat the entire argument for the grid Ω^1 which is also perfect by Proposition 5.5. We claim that for the grid Ω^1 the only possible case is again the Case 2.2. Indeed, the grid Ω^1 cannot be of types E_1 or E_2 because otherwise by Propositions 5.8, 5.9 the grid Ω itself is of type E_1 or E_2 . Therefore, we can uniquely determine $[04]$ and then $[14]$ and then all the rest

$$[ij], \quad i = 0, 1, \quad j \geq 5.$$

in the same manner.

Now, this implies that our grid Ω is one of the types I–IV. Namely, we shall check below in Proposition 6.1 that for any perfect grid Ω which is not of the elementary types E_1 or E_2 one can find a grid $\tilde{\Omega}$ of one of types I–IV such that

$$[00] = \tilde{[00]}, \quad [10] = \tilde{[10]}, \quad [01] = \tilde{[01]}, \quad [11] = \tilde{[11]}, \quad [02] = \tilde{[02]}.$$

But then, since those five values uniquely determine the rest, we conclude that $\Omega = \tilde{\Omega}$.

Therefore, to finish the proof of the theorem in the $n = 1$ case it suffices to establish the following

Proposition 6.1. *Denote by Σ the following subset of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$*

$$\Sigma = \{(0, 0), (1, 0), (0, 1), (1, 1), (0, 2), (0, 3)\} \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}.$$

Then any perfect grid Ω belongs to one of the following classes (compare with classification in Appendix A):

E₁. $[0j] = [1j]$, $j = 0, 1, 2, \dots$

E₂. $[0, j] = [1, j + 1]$, $j = 0, 1, 2, \dots$

I. *There exist $a, b, c, q, t \in \overline{\mathbb{k}}$, $q, t \neq 0, \pm 1$ such that*

$$[ij] = a + bq^j t^i + \frac{c}{q^j t^i}, \quad \forall (i, j) \in \Sigma.$$

The elements $a, b, c, q, t \in \overline{\mathbb{k}}$ are determined uniquely up to the following symmetry

$$q \mapsto q^{-1}, \quad t \mapsto t^{-1}, \quad b \mapsto c, \quad c \mapsto b.$$

II. *There exist unique $\alpha, \beta, \beta', \gamma \in \overline{\mathbb{k}}$ such that*

$$[ij] = \alpha + \beta j + \beta' i + \gamma(\beta j + \beta' i)^2, \quad \forall (i, j) \in \Sigma.$$

IIIa. *There exist unique $\alpha, \alpha', \beta, \beta' \in \mathbb{k}$ such that*

$$[ij] = \alpha + (-1)^j(\alpha' + \beta j + \beta' i), \quad \forall (i, j) \in \Sigma.$$

IIIb. *There exist $\alpha, \alpha', \beta, \beta' \in \mathbb{k}$ such that*

$$[ij] = \alpha + (-1)^i(\alpha' + \beta j + \beta' i), \quad \forall (i, j) \in \Sigma.$$

In this case only the numbers $\alpha + \alpha'$, $\alpha - \alpha' + \beta'$, and β are determined by the numbers $[ij]$, $(i, j) \in \Sigma$.

IIIc. *There exist unique $\alpha, \alpha', \beta, \beta' \in \mathbb{k}$ such that*

$$[ij] = \alpha + (-1)^{i+j}(\alpha' + \beta j + \beta' i), \quad \forall (i, j) \in \Sigma.$$

IV. *There exist unique $\alpha, \beta, \beta', q \in \overline{\mathbb{k}}$ such that*

$$\begin{aligned} [0j] &= \alpha + \beta q^j, \quad j = 0, \dots, 3, \\ [1j] &= \alpha + \beta' q^{-j}, \quad j = 0, 1. \end{aligned}$$

The proof will be based on the following lemma which can be established by direct inspection

Lemma 6.2. *For any 4-tuple*

$$(w_0, w_1, w_2, w_3) \in \mathbb{k}^4$$

satisfying

$$w_1 \neq w_2$$

we have the 3 following mutually exclusive possibilities:

(i) *There exist $a, b, c, q \in \overline{\mathbb{k}}$, $q \neq 0, \pm 1$ such that*

$$w_j = a + bq^j + \frac{c}{q^j}, \quad j = 0, \dots, 3.$$

The elements $a, b, c, q \in \overline{\mathbb{k}}$ are determined uniquely up to the following symmetry

$$q \mapsto q^{-1}, \quad b \mapsto c, \quad c \mapsto b.$$

In this case

$$w_0 - 3w_1 + 3w_2 - w_3 \neq 0 \quad \text{and} \quad w_0 + w_1 - w_2 - w_3 \neq 0.$$

(ii) $w_0 - 3w_1 + 3w_2 - w_3 = 0$ and there exist unique $\alpha, \beta, \gamma \in \mathbb{k}$ such that

$$w_j = \alpha + \beta j + \gamma \beta^2 j^2, \quad j = 0, \dots, 3.$$

(iii) $w_0 + w_1 - w_2 - w_3 = 0$ and there exist unique $\alpha, \alpha', \beta \in \mathbb{k}$ such that

$$[ij] = \alpha + (-1)^j (\alpha' + \beta j), \quad j = 0, \dots, 3.$$

The last case is a subcase of second one if $\text{char } \mathbb{k} = 2$.

Proof of Proposition 6.1. Assume that we are not in the elementary cases E_1 or E_2 and apply Lemma 6.2 to the following 4-tuple

$$[00], [01], [02], [03] \in \mathbb{k}.$$

Recall that by non-degeneracy we have $[01] \neq [02]$. Therefore we have 3 mutually exclusive possibilities which we shall call cases (i), (ii), and (iii) respectively.

(i) In this case we have

$$(6.5) \quad [0j] = a + bq^j + \frac{c}{q^j}, \quad j = 0, \dots, 3.$$

Assume for a moment that $bc \neq 0$. Choose $t_0 \in \overline{\mathbb{k}}$ so that

$$(6.6) \quad [10] = a + bt_0 + \frac{c}{t_0}.$$

Then the other root of this equation equals $\frac{c}{bt_0}$. Since we exclude the elementary cases the following equation is satisfied and is not identically zero

$$(6.7) \quad [03]G_1([10], [01], [11]) - G_0([00], [10], [01], [11], [02]) = 0.$$

Substituting (6.5) and (6.6) into (6.7) we obtain

$$(q^3b - c) \left(a + bq t_0 + \frac{c}{q t_0} - [11] \right) \left(a + \frac{bt_0}{q} + \frac{cq}{t_0} - [11] \right) = 0.$$

Since $[01] \neq [02]$ we have $bq^3 \neq c$. Hence either

$$[11] = a + bqt_0 + \frac{c}{qt_0},$$

in which case we are done by setting $t = t_0$, or

$$[11] = a + \frac{bt_0}{q} + \frac{cq}{t_0},$$

in which case we set $t = \frac{c}{bt_0}$.

Now consider the case $bc = 0$. We can assume that $c = 0$. Set

$$t = \frac{[10] - a}{b}.$$

Then the equation (6.7) implies that either

$$[11] = a + bqt,$$

which means we are in case I, or

$$[11] = a + \frac{bt}{q},$$

which brings us in the case IV.

(ii) In this case we have

$$(6.8) \quad [0j] = \alpha + \beta j + \gamma \beta^2 j^2, \quad j = 0, \dots, 3.$$

By non-degeneracy we have $\beta \neq 0$. Again, assume for a moment that $\gamma \neq 0$. Choose $\beta'_0 \in \overline{\mathbb{k}}$ so that

$$(6.9) \quad [10] = \alpha + \beta'_0 + \gamma \beta'^2_0.$$

Then the other root of this equation equals $-\beta'_0 - \frac{1}{\gamma}$. Solving the equation (6.7) for $[11]$ we obtain that either

$$[11] = \alpha + \beta + \beta'_0 + \gamma(\beta + \beta'_0)^2,$$

in which case we set $\beta' = \beta'_0$, or

$$[11] = \alpha - \beta + \beta'_0 + \gamma(-\beta + \beta'_0)^2,$$

in which case we set $\beta' = -\beta'_0 - \frac{1}{\gamma}$ to obtain

$$[11] = \alpha + \beta + \beta' + \gamma(\beta + \beta')^2.$$

Now if $\gamma = 0$ we set $\beta' = [10] - \alpha$ and the two possibilities for $[11]$ bring us in the cases II and IIIb respectively.

Similarly, the consideration of the (iii) case leads to cases IIIa and IIIc. This concludes the proof. \square

7. PROOF OF THE THEOREM FOR $n > 1$

Suppose Ω is not of type E_2 ; then by Proposition 5.8 we can assume

$$(7.1) \quad [00] \neq [11].$$

Consider the condition

$$(7.2) \quad P_{(3,0,0)}(\widehat{(2,1,1)}; \Omega) = 0.$$

Using the substitution (5.11) one computes

$$(7.3) \quad \begin{aligned} P_{(3,0,0)}(\widehat{(2,1,1)}; \Omega) &= \\ &= \varepsilon([11] - [00]) ([20] G_2([00], [10], [01], [11], [02]) - G_3([00], [10], [01], [11], [02])) \end{aligned}$$

where

$$\varepsilon = \frac{([02] - [01])([21] - [20])([11] - [20])}{([20] - [01])([10] - [02])([10] - [01])^2}$$

and G_2 and G_3 are the following polynomials

$$(7.4) \quad \begin{aligned} G_2(y_1, y_2, y_3, y_4, y_5) &:= y_4y_5 - y_4y_3 + y_2y_1 - y_2y_3 - y_1y_5 + y_3^2, \\ G_3(y_1, y_2, y_3, y_4, y_5) &:= y_4^3 - y_4^2y_2 - y_4^2y_1 - y_4^2y_3 - y_4y_2^2 + y_4y_2y_1 + y_4y_2y_5 + \\ &\quad 2y_4y_2y_3 + y_4y_1y_3 - y_4y_3^2 + y_2^3 - y_2^2y_5 - y_2^2y_3 + y_2y_5y_3 - y_2y_3^2 - y_1y_3y_5 + y_3^3. \end{aligned}$$

Recall that we assume that $[00] \neq [11]$. Therefore, from (7.2) we have a linear equation in $[20]$ and hence two possible cases:

Case 1. We have

$$G_2 = G_3 = 0.$$

Using (7.1) we can express $[02]$ from $G_2 = 0$

$$(7.5) \quad [02] = \frac{[01]^2 - [10][01] - [11][01] + [10][00]}{[00] - [11]}.$$

Substituting this expression into the equation $G_3 = 0$ we obtain

$$(7.6) \quad \varepsilon([11] - [01])([11] - [00] - [01] + [10])([11] - [00] + [01] - [10]) = 0,$$

where $\varepsilon = [11] - [10]$. Observe that it is impossible to have $[11] = [01]$ because then from (7.5) we conclude $[02] = [10]$, which contradicts non-degeneracy. Similarly, it is impossible to have

$$[11] = [00] + [01] - [10]$$

because then from (7.5) we conclude

$$[02] = [00]$$

again in contradiction with non-degeneracy. Thus, the only possible solution of (7.6) and (7.5) is

$$\begin{aligned} [11] &= [00] - [01] + [10], \\ [02] &= 2[01] - [00]. \end{aligned}$$

That is, the grid Ω_2 is of the type IIIb. Recall that the parameters $\alpha, \alpha', \beta, \beta'$ are not uniquely determined by the grid Ω_2 (see Proposition 6.1). We can find such values of $\alpha, \alpha', \beta, \beta' \in \mathbb{k}$ that

$$[ij] = \alpha + (-1)^i(\alpha' + \beta j + \beta' i), \quad i + j \leq 2.$$

for all $i + j \leq 2$. Then by (5.6) we conclude that

$$(7.7) \quad [ij] = \alpha + (-1)^i(\alpha' + \beta j + \beta' i), \quad \forall i \leq 2 \quad \forall j \geq 0.$$

Note that now the parameters $\alpha, \alpha', \beta, \beta'$ are uniquely determined. If $n = 2$ this finishes the consideration of the Case 1.

If $n > 3$ then we have to look at the equation

$$(7.8) \quad P_{(2,1,0,0)}(\widehat{(1,1,1,1)}) = 0.$$

By (7.7) one concludes from the identity

$$F(u, u + z, v, v - z, w) = w + z$$

and (5.6) that

$$(7.9) \quad [31] = [30] - \beta.$$

From (7.7) and (7.9) one computes that (7.8) is equivalent to

$$(\alpha - \alpha' - 3\beta' - [30])([21] - [30]) = 0$$

By non-degeneracy we conclude that

$$[30] = \alpha - \alpha' - 3\beta'$$

and then by (5.6) and (7.7) it follows that

$$[3j] = \alpha - \alpha' - 3\beta' - \beta j.$$

Thus, the grid Ω is of the type IIIb. Repeating the above argument for the grids ${}_1\Omega, {}_2\Omega, \dots$ we conclude by Proposition 5.5 that the entire grid Ω is of the type IIIb.

Case 2. Now we assume that Ω is not of the IIIb type and then we can determine $[20]$ from (7.3)

$$(7.10) \quad [20] = \frac{G_3([00], [10], [01], [11], [02])}{G_2([00], [10], [01], [11], [02])}.$$

After that using (5.6) we can also determine $[21], [22], \dots$. It is clear that if the grid Ω_1 is of the type E₁, I, II, IIIa, IIIc then the grid Ω_2 is of the same type with the same parameters. If the grid Ω_1 is of type IV then

$$G_2 = 0, \quad G_3 \neq 0,$$

that is, the equation (7.2) has no solutions. For $n = 2$ this concludes the proof.

If $n > 2$ then we have to repeat the entire argument for the grids ${}_1\Omega, {}_2\Omega, \dots$. In order to be able to do so we have to verify that once the grid Ω_2 is not of type IIIb then neither is the grid ${}_1\Omega_3$. But this follows from the just established classification of the perfect grids for $n = 2$. Namely, it follows that if Ω is a perfect grid then

$$({}_1\Omega_2 \text{ is of type IIIb}) \implies (\Omega_2 \text{ is of type IIIb}).$$

This concludes the proof of the theorem.

APPENDIX A. TABLE OF PERFECT GRIDS.

E₁. The first elementary case

$$[ij] = \gamma_j,$$

where $\gamma_0, \gamma_1, \dots$ are arbitrary pairwise distinct. The corresponding interpolation polynomials are factorial monomial symmetric functions.

E₂. The second elementary case

$$[ij] = \gamma_{j-i},$$

where $\dots, \gamma_{-1}, \gamma_0, \gamma_1, \dots$ are arbitrary pairwise distinct. The corresponding interpolation polynomials are factorial Schur functions.

I. The generic case of Macdonald interpolation polynomials

$$[ij] = a + bq^j t^i + \frac{c}{q^j t^i},$$

where $q, t \neq 0$. The non-degeneracy conditions are

$$\begin{aligned} q^k &\neq t^l, & \forall k, l \quad k > 0, 0 \leq l \leq n, \\ bq^k t^l &\neq c, & \forall k, l \quad k > 0, 0 \leq l \leq 2n. \end{aligned}$$

If $t = 1$ or $t = 1/q$ we hit the two elementary cases above.

II. The case

$$[ij] = \alpha + \beta j + \beta' i + \gamma(\beta j + \beta' i)^2$$

can be obtained from I by setting

$$\begin{aligned} a &= \alpha - 2\frac{\gamma}{h^2}, & b &= \frac{1}{2h} + \frac{\gamma}{h^2}, & c &= -\frac{1}{2h} + \frac{\gamma}{h^2}, \\ q &= 1 + \beta h, & t &= 1 + \beta' h \end{aligned}$$

and letting $h \rightarrow 0$. The non-degeneracy conditions are

$$\begin{aligned} k\beta \neq l\beta', & \quad \forall k, l \quad k > 0, 0 \leq l \leq n, \\ \gamma(k\beta + l\beta') \neq -1, & \quad \forall k, l \quad k > 0, 0 \leq l \leq 2n. \end{aligned}$$

The cases II and III(abc) are impossible if $\text{char } \mathbb{k} > 0$.

III(abc). The case

$$[ij] = \alpha + \epsilon^j \epsilon'^i (\alpha' + \beta j + \beta' i)$$

can be obtained from I by setting

$$\begin{aligned} a &= \alpha, & b &= \frac{\alpha'}{2} + \frac{1}{2h}, & c &= \frac{\alpha'}{2} - \frac{1}{2h}, \\ q &= \epsilon(1 + \beta h), & t &= \epsilon'(1 + \beta' h) \end{aligned}$$

and letting $h \rightarrow 0$. Here $\epsilon, \epsilon' = \pm 1$; namely, we have three subcases: **(a)** $(\epsilon, \epsilon') = (-1, 1)$, **(b)** $(\epsilon, \epsilon') = (1, -1)$, **(c)** $(\epsilon, \epsilon') = (-1, -1)$.

IV. This case exists only for $n = 2$ when we can have

$$[0j] = \alpha + \beta q^j, \quad [1j] = \alpha + \beta' q^{-j}.$$

It is obtained from I by setting

$$a = \alpha, \quad b = \beta, \quad c = \beta' t$$

and then letting $t \rightarrow 0$. The non-degeneracy in this case is equivalent to $\beta, \beta' \neq 0$ and $q^k \neq 1, \beta'/\beta$ for all $k > 0$.

APPENDIX B. AN EXAMPLE OF A UNIVERSAL INTERPOLATION POLYNOMIAL.

Consider the case $n = 1$ and

$$\mu = (3, 0).$$

Then the universal polynomial $P_\mu^u(x_0, x_1)$ lies in

$$P_\mu^u(x_0, x_1) \in \mathbb{Z} \left[u_{00}, u_{01}, u_{02}, u_{10}, u_{11}, \frac{1}{u_{02} - u_{10}}, \frac{1}{u_{01} - u_{10}} \right] [x_0, x_1].$$

In the basis of the monomial symmetric functions m_λ it is given by the formula

$$P_\mu^u(x_0, x_1) = m_{30} + \frac{c_{21}(u) m_{21} + c_{20}(u) m_{20} + c_{11}(u) m_{11} + c_{10}(u) m_{10} + c_{00}(u)}{(u_{02} - u_{10})(u_{01} - u_{10})},$$

where c_{21}, \dots, c_{00} are the following polynomials

$$\begin{aligned} c_{21} = & u_{02}u_{00} - u_{02}u_{10} + u_{02}u_{01} - u_{02}u_{11} - u_{00}u_{10} + u_{00}u_{01} - u_{00}u_{11} + u_{10}^2 \\ & - u_{10}u_{01} + u_{10}u_{11} - u_{01}u_{11} + u_{11}^2, \\ c_{20} = & u_{02}^2u_{10} - u_{02}^2u_{01} - u_{02}u_{00}u_{01} + u_{02}u_{10}u_{01} + u_{02}u_{10}u_{11} - u_{02}u_{01}^2 + u_{00}u_{10}u_{11} \\ & - u_{10}^3 - u_{10}^2u_{11} + u_{10}u_{01}^2 + u_{10}u_{01}u_{11} - u_{10}u_{11}^2, \\ c_{11} = & -(u_{02} + u_{10} + u_{01} + u_{11})(u_{02}u_{00} - u_{02}u_{10} + u_{02}u_{01} - u_{02}u_{11} - u_{00}u_{10} \\ & + u_{00}u_{01} - u_{00}u_{11} + u_{10}^2 - u_{10}u_{01} + u_{10}u_{11} - u_{01}u_{11} + u_{11}^2), \\ c_{10} = & u_{02}^2u_{00}u_{01} - u_{02}^2u_{10}^2 - u_{02}^2u_{10}u_{11} + u_{02}^2u_{01}^2 + u_{02}u_{00}u_{01}^2 + u_{02}u_{10}^3 - u_{02}u_{10}^2u_{01} \\ & - u_{02}u_{10}u_{01}u_{11} - u_{00}u_{10}^2u_{11} - u_{00}u_{10}u_{11}^2 + u_{10}^3u_{01} + u_{10}^3u_{11} - u_{10}^2u_{01}^2 \\ & + u_{10}^2u_{11}^2 - u_{10}u_{01}^2u_{11} + u_{10}u_{11}^3, \\ c_{00} = & -u_{02}^2u_{00}u_{01}^2 + u_{02}^2u_{10}^2u_{01} + u_{02}^2u_{10}^2u_{11} - u_{02}^2u_{10}u_{01}^2 - u_{02}u_{10}^3u_{01} - u_{02}u_{10}^3u_{11} \\ & + u_{02}u_{10}^2u_{01}^2 + u_{02}u_{10}^2u_{01}u_{11} + u_{00}u_{10}^2u_{11}^2 - u_{10}^3u_{01}u_{11} + u_{10}^2u_{01}^2u_{11} - u_{10}^2u_{11}^3. \end{aligned}$$

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